

Examination of Mr. Whittaker's "Undulatory Explanation of Gravity" from the Physical Standpoint. By G. Johnstone Stoney, M.A., Hon. Sc.D., F.R.S.,

In the part of the *Monthly Notices* issued last September, at p. 619 of vol. lxii., there is a suggestive paper by Mr. Edmund T. Whittaker dealing with three topics.

1. In the first section of his paper he gives a striking and most useful general solution of Laplace's Differential Equation for the potential of gravity, viz. of the equation

$$\nabla^2 V = 0, \text{ or a constant.} \quad \dots \quad \dots \quad (1)$$

2. In the second section by a similar method of treatment he obtains a singularly elegant proof of the recently discovered theorem that the wave-motion represented by the equation

$$\nabla^2 V = \frac{1}{k^2} \frac{\partial^2 V}{\partial t^2} \quad \dots \quad \dots \quad \dots \quad (2)$$

(in other words, wave-motion which is propagated with constant speed k in all directions through an isotropic medium) can be analysed into components, each of which is an undulation of flat wavelets. This is the principal case of a more general theorem which effects the resolution in crystalline as well as isotropic media; and this proof in the case of isotropic media,* for which we are indebted to Mr. Whittaker, has deduced the resolution from the fundamental differential equation of such motion, and is therefore a most welcome addition to the previously known proofs of the theorem.

3. In the last section of his paper Mr. Whittaker develops "An undulatory explanation of gravity" which aims at "analysing the field of force [? acceleration †] due to a gravitating body by a 'Spectrum Analysis,' as it were, into an infinite number of constituent fields . . . of an undulatory character." This rather

* It seems likely that Mr. Whittaker's remarkable handling of equations (1) and (2) could present the general solution of equation (2) as an integral of elements each of which is an undulation of flat wavelets, in the case where k is a function of the polar coordinates θ and ϕ , as well as in the case dealt with by Mr. Whittaker where k is a constant in reference to θ , ϕ , and r . It would be of much interest to ascertain whether this is so, but the present writer has not gone sufficiently into this question. If so, Mr. Whittaker's analysis will extend to crystalline as well as isotropic media, just as the method of proof by the Principle of Reversal does (see *British Association Report* for 1901, p. 570). It may be desirable to add that the proof by the method of Reversal supplies further information which is of use in making experiments, inasmuch as it exhibits the relation in which the resolution into flat wavelets stands to resolutions into wavelets of other forms. It seems probable that this information could not be obtained without much difficulty by the analytical method.

† See "How the Misuse of the Word *Force* may be avoided," *British Association Report* for 1894, p. 586.

startling announcement seems to require further discussion than that given to it by its author, in order that we may understand what the proposed resolution means *physically* and what it does not mean. The physical meaning of his equations is not referred to in Mr. Whittaker's paper, and an inquiry into it is therefore undertaken here, since without some knowledge about it we are not in a position to form a judgment as to the true value of the resolution.

4. In the department of physics to which Mr. Whittaker's investigation belongs a differential equation has first to be so constructed as to express the fact that physical causes, acting under definite laws, are in operation; and having obtained this differential equation by a study of the laws under which the physical causes act in the problem of nature which is under investigation, then the innumerable particular solutions of that differential equation will represent the several events in nature which those physical causes are competent to produce.

5. Thus, in acoustics, the differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{k^2} \frac{\partial^2 y}{\partial t^2}$$

represents the physical causes that govern the transverse vibrations of a uniform flexible and elastic string when stretched and set vibrating, so long as the range of vibration is small; and the fundamental note emitted by this string and all its harmonics present themselves as motions represented by particular solutions of this equation. In this and all analogous cases the sum of any of these particular solutions will also be a solution of the differential equation; the physical meaning of which is that any group of the harmonics, or of the harmonics and fundamental tone, produce a complex resultant tone of which the members of that group are true physical components.

A similar state of things prevails when we investigate mathematically that spectrum analysis of light to which Mr. Whittaker likens his analysis of gravity. Each of the luminous undulations is a particular solution of the differential equation which is to be employed when dealing with light; and *any* group of these particular solutions when added together represents a resultant luminous effect of which the members of that group are genuine physical components.

The same is also true of that other resolution of light into its flat wavelet components which is dealt with by Mr. Whittaker in the second part of his paper. Here, again, the resultant is the sum of the components, and the components and resultant are all of them solutions of the fundamental differential equation: in other words they are, whether taken separately or as coexisting in any group, effects which the physical causes represented by the differential equation are competent to produce.

The same holds true of fields of potential due to gravitation.

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Each attracting centre produces its own field of potential, and the coexistence of any number and any disposition of attracting centres produces a resultant field which is represented mathematically as the sum of the component fields.

6. All these are cases in which, by applying an unrestricted treatment to the differential equation, we obtain true physical resolutions; but the resolution of gravity suggested by Mr. Whittaker is of a different kind. In the first place it does not result from an unrestricted treatment of the differential equation, since it is only by picking out one particular and very special solution of the equation which Mr. Whittaker suggests shall be substituted for Laplace's equation, that we can arrive at his resolution; and it only remains to inquire what is the physical meaning of the process by which this artificial resolution is obtained.

7. To convert Mr. Whittaker's analysis into a form which will exhibit its physical meaning, it will suffice to differentiate his equations by r , and thus transform them from equations that refer to potential into equations that refer to accelerations. The definite case with which Mr. Whittaker deals is that of the field of potential called into existence by an attracting centre situated at the origin of coordinates. In this case his undulatory components assume the form

$$dV = \frac{2d\mu}{\pi\mu} \cdot \frac{\sin \mu(kt-r)}{r} \quad \dots \quad \dots \quad (3)$$

where μ is the wave-frequency and k the speed of propagation of the undulation. Differentiating this equation by r we obtain the following expression for the corresponding component of the attracting acceleration:—

$$dR = -\frac{2d\mu}{\pi} \left(\frac{\sin \mu(kt-r)}{\mu} \cdot \frac{1}{r^2} + \cos \mu(kt-r) \cdot \frac{1}{r} \right) \dots \quad (4)$$

This shows that this component of the field of acceleration consists of two attractions at each distance r from the origin, one an acceleration varying inversely as the distance, the other an acceleration varying inversely as the square of the distance, and both multiplied by coefficients which make their intensity at each point of space fluctuate periodically with the frequency μ .

8. Accordingly the general outcome is that we are to imagine two infinite series of accelerations the terms of which correspond to all possible values of μ , one series consisting of accelerations varying inversely as the distance, the other of accelerations varying inversely as the square of the distance, and all these accelerations flickering in intensity at each point of space in definite ways. Then what the physicist learns from Mr. Whittaker's analysis is that *there is one definite set of coefficients* which if prefixed to these innumerable flickering accelerations will enable them to furnish an unvarying resultant at each point of space.

This is an effect of a kind which it is antecedently probable could by suitable manipulation be brought about ; but unfortunately results obtained by such manipulation, though pleasing from their ingenuity, do not hold out any prospect of being of service to the student of dynamical astronomy.

9. Viewed from the mathematical standpoint, Mr. Whittaker's investigation amounts to this, that when we integrate equation (3) we find in it an instance of a state of things occasionally met with—viz. that where a variable, such as r in equation (3), presents itself both under and outside a sign of integration, then this variable as it exists outside the sign of integration will alone survive when the integration is taken between definite limits. This, for example, will happen whenever, as in the present instance, viz., in—

$$V = \frac{2}{\pi r} \int \frac{\sin \mu(kt-r)}{\mu} d\mu \quad \dots \quad \dots \quad (3a)$$

or

$$V = \frac{2}{\pi r} \int \frac{\sin \mu(kt-r)}{\mu(kt-r)} d[\mu(kt-r)] \quad \dots \quad \dots \quad (3b)$$

the variable in question only appears under the sign of integration in a form in which it can be presented as part of a coefficient to the variable by which we are integrating, and when the limits of integration are then made cipher and infinity. In Mr. Whittaker's integral the transversal (or square root of the intensity) of each element of the series represented by the integral is assumed to be proportional to $1/\mu$, the wave-length, and when this very special set of values are assigned to the coefficients of the terms of the series represented by the integral, then r under the sign of integration, which is essential to the physical interpretation, is excluded from the resultant while appearing in the components. This produces the semblance of a resolution notwithstanding that in the components on the one hand, and in the resultant on the other, we are dealing mathematically with differential equations—equations (1) and (2)—the general solutions of which are not physically connected. What Mr. Whittaker has shown is that one particular solution of equation (2) can be obtained by a definite integration so contrived as to obliterate *in that particular solution* the physical incompatibility which in general prevails between equations (1) and (2). This would, possibly, not be an altogether illegitimate process if the physical events which would justify it were such as we might assume to prevail in actual nature. The physical assumption to be made is that the attraction of each particle of ponderable matter can be of that extraordinary kind described above in paragraphs (7) and (8). But, unfortunately, the improbability of this assumption is so overwhelming that no *physicist* can feel himself justified in seriously arguing from the supposition that it represents the state of things which exists in nature, however

much he may be impressed by the mathematical ingenuity of the investigation.

Postscript.

A discussion followed the reading of this paper in which it was pointed out that there exist motions in the solar system which cannot be wholly accounted for by gravitation acting in accordance with Laplace's Differential Equation; and it was suggested that this indicates that Laplace's equation requires modification, at least for bodies in relative motion. Now if Laplace's equation has to be modified it appears certain from the physical interpretation of Mr. Whittaker's equations that the modification cannot be that substitution of equation (2) for Laplace's equation which is suggested by Mr. Whittaker; and that what we learn from the investigation is that we may limit the inquiry by dismissing this alternative when searching for the modification.

But has it been quite ascertained that the observed motions may not be due to causes already known? We are acquainted with at least three causes acting within the solar system which *must* produce deviations from Newtonian motions. The chief of these is the meteoric streams and sporadic meteorites which dash about in vast swarms within the space through which the visible bodies of the solar system have to travel. Another such cause is the escape of gases from atmospheres (see *Astrophysical Journal* for June 1900, Appendix, p. 369). This escape we have reason to believe takes place chiefly from the side of a planet turned towards the Sun, and therefore produces on the planet a one-sided effect; and a third cause which is known to exist is electric. Modern astronomers are able to compute perturbations with such refinement that it does not seem impossible that the first of the above *veræ causæ* may produce effects of sufficient amount to be detected.

It is noteworthy that at least the first two of these *veræ causæ* would operate with more intensity upon *Mercury* than upon our Earth: witness much other evidence, corroborated by the existence of such phenomena as the Solar Corona and the Zodiacal Light.

Note on the Present Condition of the Lunar Theory.

By E. Nevill.

1. The recent discussion at the March Meeting of the Royal Astronomical Society on Professor Newcomb's note, entitled "On the desirableness of a reinvestigation of the problems growing out of the Mean Motion of the Moon," suggests my furnishing some information with respect to the researches on